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On the Periodic Oscillations of $\ddot{x} = g(x)$

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In this paper we introduce certain generalized concepts of isochronism for the periodic oscillations of the scalar equation $\ddot{x} = g(x)$. We give characterizations of these concepts; we also study two such characterizations, as well as some related topics, in more depth. Each one of these two last concepts is equivalent to the stability of the equilibrium for some related system of differential equations. © 1989 Academic Press, Inc.

1. INTRODUCTION

To introduce our problem let us first recall some previous results.

In the paper [16], which appeared in this Journal, the author determined and constructed the family of all the real continuous maps f such that the origin is a stable equilibrium point for the system

$$\ddot{x} + xf(x) = 0, \ \ddot{y} + yf(x) = 0, \ (x, y) \in \mathbb{R}^2, \ f(0) > 0 \tag{1.1}$$

where f is defined in some neighbourhood of 0 in $\mathbb{R}$.

The condition $f(0) > 0$ implies that the origin is a center for the first equation $\ddot{x} + xf(x) = 0$ and it is necessary for the stability of the origin for the full system (1.1) (as can be seen by considering $f(0) \leqslant 0$ and the solutions along the y axis).

From the mechanical point of view, (1.1) is related to a purely positional force which is central and nonconservative (unless f is constant). Also, $f(0) > 0$ says that the force is attractive locally at the origin. In spite of the attractiveness of the force, the origin is proved to be generically unstable. But there are many f that yield stability. In fact, for any given continuous function $\bar{f}$: $[0, a[\to \mathbb{R}$, with $\bar{f}(0) > 0$, $a > 0$, there exists a continuous extension f of $\bar{f}$ to some neighbourhood of 0 which yields stability of the origin for (1.1). Moreover, this extension is unique up to the equivalence which defines the C^0 germs at 0; see [16].

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We are also interested in the following one-parameter family of systems

$$\ddot{x} + xf(x) = 0, \qquad \ddot{y} + y\frac{f(x) + xf'(x) + \alpha x(4f(x) + xf'(x))}{1 + \alpha x} = 0, \qquad f(0) > 0, \tag{1.2}$$

where $\alpha \in \mathbb{R}$ is the parameter, and now $f \in C^1$ is required. By setting $g(x) = -xf(x)$, this system looks simpler, namely

$$\ddot{x} = g(x), \qquad \ddot{y} = y\left(g'(x) + \frac{3\alpha g(x)}{1 + \alpha x}\right), \qquad g'(0) < 0. \tag{1.3}$$

The problem of finding the family of all the C^1 maps f such that the origin is stable for (1.2) was solved in [18]. The result is similar to the preceding one. Namely $f(0) > 0$ is necessary for stability, and we can arbitrarily give $\bar{f} \in C^1([0, a[; \mathbb{R})$, with $a > 0$ and $\bar{f}(0) > 0$. This map determines a unique C^1 germ which yields stability and agrees with $\bar{f}$.

The system (1.2) was found in [17] by looking for problems which "have the same nature" as the one related to (1.1) (see [17] and the introduction of [18]).

Let us consider (1.3) with $\alpha = 0$. If we plug a solution $x(\cdot)$ of $\ddot{x} = g(x)$ into $\ddot{y} = yg'(x)$, then this last equation becomes the linear variational equation associated with $\ddot{x} = g(x)$ and its solution $x(\cdot)$. It is easy to see that Liapunov stability of the origin for (1.3) with $\alpha = 0$ is equivalent to the (local) isochronism of the oscillations of the first equation $\ddot{x} = g(x)$ (near the origin of the $x, \dot{x}$-plane). Are there some similar properties in connection with (1.3) with $\alpha \neq 0$, and with (1.1)?

In order to answer this question let us first introduce a general definition.

Consider the scalar equation (in some neighbourhood of the origin)

$$\ddot{x} + xf(x) = 0, \qquad f(0) > 0, f \in C^0, \tag{1.4}$$

and a strictly positive map m defined and continuous in some neighbourhood of 0 in $\mathbb{R}$. Let $x(\cdot, x_0, \dot{x}_0)$ be a periodic solution of (1.4) which stays in the domain of m and has $(x_0, \dot{x}_0)$ as initial condition. Then we consider the "artificial time"

$$\tau(t, x_0, \dot{x}_0) = \int_0^t m(x(\xi, x_0, \dot{x}_0))\, d\xi, \tag{1.5}$$

which is obtained by the "weight" m. We say that the origin is (locally) a *m-isochronous* center for (1.4) iff we have (local) isochronism with respect to the "artificial time" in (1.5) (see Def. 1 in Sect. 2). In this case we use the notation:

$$\oint m(x)\, dt = \text{const. (locally)}. \tag{1.6}$$

In Section 5 we answer the aforementioned question. Namely we see that the origin is a stable equilibrium for (1.2) iff $f(0)>0$ and

$$\oint (1+\alpha x)^{-2}\, dt = \text{const. (locally)} \tag{1.7}$$

for the equation $(1.2)_1$ (which coincides with (1.4) but now $f \in C^1$). Furthermore, Liapunov stability of the origin for (1.1) is equivalent to $f(0)>0$ and

$$\oint f(x)\, dt = \text{const. (locally)}. \tag{1.8}$$

However we firstly deal with the general m-isochronism. In Section 2 we characterize m-isochronism in terms of the maps u and h already considered in our [16] and [18]. These maps play a fundamental role in this paper too. So Section 2 starts recalling their definitions.

In Section 3 we study f-isochronism and $(1+\alpha x)^{-2}$-isochronism. We also see that, whenever both these properties hold, we simply have

$$f(x) = f(0)/(1+\alpha x)^3. \tag{1.9}$$

We call this function the "*Keplerian map*" for reasons that will be clear in Section 4. In Section 4 we also see that condition (1.7) has a direct *physical meaning* (for suitable $\alpha>0$) in connection with the equation of the radial motion in a central potential field.

Let us remark that (1.9) was found by Barone and Cesar in [1]. They proved that (1.9) yields stability of the origin for (1.1) via an interesting argument which is not related to the Kepler's problem but to the existence of "energy-like" Liapunov functions for the system (1.1) which is not conservative.

In Section 6 we consider some simple constructions of the centers satisfying (1.7) and we do the same for (1.8). We also show a geometrical construction in the case of (ordinary) isochronism in terms of a translating parabola. (We could say that this construction introduces an ideal pantograph for the isochronous centers).

Also the last section (Sect. 7) considers the particular case of the isochronous centers. Its title is: "Oscillations with prescribed periods".

Ordinary isochronism for equation (1.4) was already studied by several authors. This is a particular case in our contest, however, also in this case the author hopes that this paper says something new. The approach in terms of the maps u and h, already considered in our [16] and [18], leads to simple formulas which permit, in particular, the aforementioned geometrical construction and the arguments in Section 7.

The first paper on (ordinary) isochronism was probably [2] by Koukles and Piskounov (see [2] or p. 65 in [14]).

In Paragraph 12 of Landau and Lifshitz's book [3] there is a nice sketch entitled "Determination of the Potential Energy from the Period of Oscillation".

In [10], Urabe gives a necessary and sufficient condition for isochronism, in the analytic case, which is fairly simple. His proof was soon extended to much weaker assumptions by Levin and Schatz in [4], and, independently, by Urabe in his second paper [11] on this subject. In that paper he gives also a solution to the more general problem of the oscillations with prescribed periods. In [12], [13], [14], Urabe deeply studies these topics.

Finally, in [6], Obi presents a characterization of the isochronous centers (which involves a map ϕ corresponding to $z \mapsto (u^{-1}(z)+u^{-1}(-z))/2$ in our notations, see (2.2.1) in [6]). He also finds some expressions by which he can deal with some other questions. In particular he gives some sufficient conditions for the monotonicity of the period (as a function of the upper amplitude of the solutions).

2. m-ISOCHRONISM

Consider the scalar equation (1.4). It has the first integral of energy:

$$\dot{x}^2 + 2v(x), \text{ where } v(x) = \int_0^x f(\xi)\, \xi \, d\xi. \tag{2.1}$$

The potential energy v is C^1 and admits

$$v''(0) = \lim_{x \to 0} \frac{v'(x)}{x} = f(0) > 0. \tag{2.2}$$

Therefore v has a strict minimum at $x = 0$. By this, and by the energy conservation, we can prove the uniqueness of the solutions to (1.4). For any $(x_0, \dot{x}_0)$ in some neighbourhood of the origin, the orbit $\gamma(x_0, \dot{x}_0)$ of (1.4) through $(x_0, \dot{x}_0)$ (in the x, $\dot{x}$-plane) is periodic and enclose the origin: the origin is a center.

Let us consider the map $\bar{u}: x \mapsto (\operatorname{sgn} x)(2v(x))^{1/2}$. Since $(2v(x)/x^2) \to f(0)$ as $x \to 0$, then $\bar{u}'(0) = (f(0))^{1/2}$. Furthermore $\bar{u}'(x) = (xf(x)/\bar{u}(x)) \to \bar{u}'(0) > 0$ as $x \to 0$. Therefore there exists a maximal interval J, with $0 \in J = J^0$, such that u defined by $u = \bar{u} \mid J$ is a C^1 diffeomorphism onto a symmetric interval I:

$$u \in \mathrm{Diff}^1(J; I),\ (u(x))^2 = 2v(x),\ u'(0) = (f(0))^{1/2}. \tag{2.3}$$

We use the notation

$$X = u^{-1}: I \to J. \tag{2.4}$$

In the sequel an important role is played by u and by the map

$$h: J \to J,\ x \mapsto X(-u(x)) \text{ (thus } v(h(x)) = v(x)). \tag{2.5}$$

We have

$$h \in \mathrm{Diff}^1(J; J),\ h^{-1} = h,\ h(0) = 0,\ h'(0) = -1. \tag{2.6}$$

For any $x_0 \in J \setminus \{0\}$, the orbit $\gamma(x_0, 0)$ of (1.4) through $(x_0, 0)$ is periodic and intersects the x-axis at $h(x_0)$ too. We denote its period by $\ell(x_0)$.

Finally, let us end these preliminaries with a notation useful in the sequel (where $\bar{J}$ is the closure of J in $\mathbb{R}$).

$$J(a) = [0, a[\cup h([0, a[) \text{ for } a \in \bar{J} \cap \mathbb{R}_+^*. \tag{2.7}$$

Definition 1. Let us consider the scalar equation (1.4) (where f is a real continuous map defined in some neighbourhood of 0 in $\mathbb{R}$), let $x(\cdot, x_0, 0)$ be its solution with $(x_0, 0)$ as initial condition, and let J be as above.

For $m \in C^0(J; \mathbb{R}_+^*)$ and $x_0 \in J \setminus \{0\}$, we define

$$\not{p}_m(x_0) = \int_0^{\ell(x_0)} m(x(t, x_0, 0))\, dt, \tag{2.8}$$

where $\ell(x_0)$ is the period of $x(\cdot, x_0, 0)$. Furthermore, let $a \in \bar{J} \cap \mathbb{R}_+^*$, then we write

$$\oint m(x)\, dt = \text{const., for } x \in J(a) \text{ (see (2.7))}, \tag{2.9}$$

iff the map $]0, a[\to \mathbb{R}_+^*$, $x_0 \mapsto \not{p}_m(x_0)$ is constant.

Whenever (2.9) holds for some $a \in \bar{J} \cap \mathbb{R}_+^*$, we say that the origin is (locally) a m-isochronous center, or we use the notation in (1.6).

Proposition 1. *Let $a \in \bar{J} \cap \mathbb{R}_+^*$. The property in* (2.9) *holds if and only if*

$$2m(0)\, u(x) = (f(0))^{1/2} \int_{h(x)}^{x} m(\xi)\, d\xi, \textit{ for every } x \in J(a). \tag{2.10}$$

Before the proof let us state the following simple Proposition.

PROPOSITION 2. *The map* $p_m\colon J \to \mathbb{R}_+^*$, *defined by* (2.8) *for* $x_0 \neq 0$ *and by*

$$p_m(0) = 2\pi m(0)/(f(0))^{1/2},$$

is continuous.

(In these Propositions f and m are as in Def. 1).

Proof of Proposition 1 *and Proposition* 2. In this proof we use the notations

$$M = m \circ X \text{ and } \mathscr{P}_m = p_m \circ X \text{ (where } X\colon I \to J \text{ is } u^{-1}). \tag{2.11}$$

Remark that $\mathscr{P}_m$ is even (see the sentence following (2.6)).

By the first integral in (2.1), (2.8) yields

$$p_m(x) = 2 \int_{h(x)}^{x} (2(v(x) - v(\xi)))^{-1/2}\, m(\xi)\, d\xi, \text{ for } x \in J,\ x > 0. \tag{2.12}$$

Therefore

$$\mathscr{P}_m(z) = p_m(X(z)) = 2 \int_{-z}^{z} (z^2 - \eta^2)^{-1/2}\, M(\eta)\, X'(\eta)\, d\eta, \text{ for } z \in I,\ z > 0.$$

So

$$\mathscr{P}_m(z) = 2 \int_0^z (z^2 - \eta^2)^{-1/2}\, (M(\eta)\, X'(\eta) + M(-\eta)\, X'(-\eta))\, d\eta, \qquad \text{for} \quad z \in I, \quad z > 0. \tag{2.13}$$

This formula gives Proposition 2 by the change of variables $\eta \to \sin^{-1}(\eta/z)$, and by using $X'(0) = 1/(f(0))^{1/2}$ (see $(2.3)_3$ and (2.4)), $\mathscr{P}_m(-z) = \mathscr{P}_m(z)$, and $p_m = \mathscr{P}_m \circ u$ (see $(2.11)_2$).

Now, let us return to (2.13). The following argument is natural if we remember how one deals with Abel's integral equation (see e.g., [9]).

$$\begin{aligned}
&\int_0^z \theta(z^2 - \theta^2)^{-1/2} \left(2 \int_0^\theta (\theta^2 - \eta^2)^{-1/2}\, (M(\eta)\, X'(\eta) + M(-\eta)\, X'(-\eta))\, d\eta \right) d\theta \\
&\quad = \int_0^z \left(2 \int_\eta^z \theta(z^2 - \theta^2)^{-1/2}\, (\theta^2 - \eta^2)^{-1/2}\, d\theta \right) \\
&\qquad \times (M(\eta)\, X'(\eta) + M(-\eta)\, X'(-\eta))\, d\eta \\
&\quad = \pi \int_0^z (M(\eta)\, X'(\eta) + M(-\eta)\, X'(-\eta))\, d\eta
\end{aligned}$$

where the last equality can be obtained by the change of variables $\theta \mapsto (\theta^2 - \eta^2)/(z^2 - \eta^2)$. Therefore (2.13) implies

$$\int_0^z \theta(z^2 - \theta^2)^{-1/2} \mathscr{P}_m(\theta)\, d\theta = \pi \int_{-z}^{z} M(\eta)\, X'(\eta)\, d\eta, \text{ for every } z \in I,\ z > 0. \tag{2.14}$$

Now, let (2.9) hold. Then (2.14) yields (2.10) since

$$\mathscr{P}_m(\theta) = \mathscr{P}_m(0) = 2\pi m(0)/(f(0))^{1/2} \text{ for every } \theta \in u([0, a[),$$

(remark that the equality in (2.10) holds for every $x \in]0, a[$ iff it holds for every $x \in J(a)$).

Let us complete the proof by showing that (2.10) implies (2.9).

By (2.10) and (2.14)

$$\mathscr{P}_m(0) z = \int_0^z \theta(z^2 - \theta^2)^{-1/2} \mathscr{P}_m(\theta)\, d\theta, \text{ for every } z \in u(]0, a[).$$

Thus

$$0 = \int_0^z \eta(z^2 - \eta^2)^{-1/2} (\mathscr{P}_m(\eta) - \mathscr{P}_m(0))\, d\eta, \text{ for every } z \in u(]0, a[), \tag{2.15}$$

where $\mathscr{P}_m$ is continuous. Now, the same argument which gave (2.14) from (2.13) yields our result. We just have to replace z by θ in (2.15), to multiply by $\theta(z^2 - \theta^2)^{-1/2}$, and to integrate with respect to θ from 0 to z. This gives

$$0 = \int_0^z \eta(\mathscr{P}_m(\eta) - \mathscr{P}_m(0))\, d\eta, \text{ for every } z \in u(]0, a[).$$

This trivially gives (2.9). ∎

3. Some Choices of the "Weight" m.

In this Section we consider f and the related maps v, $u: J \to I$, h, as in Section 2. We study the condition in (2.9), for some particular choices of m, taking as understood that $a \in \bar{J} \cap \mathbb{R}_+^*$.

Let us firstly state the following noteworthy property (even if it is not used in the sequel).

Corollary 1. *For every a we have*

$$\oint u'(x)\, dt = \text{const.}, \textit{ for } x \in J(a). \tag{3.1}$$

This Corollary 1, as well as the following Corollary 2, is obtained at once from Proposition 1 in Section 2 (and from the properties of u and h discussed in the first part of Sect. 2).

COROLLARY 2. *Let $\alpha \in \mathbb{R}$. We have*

$$\oint (1+\alpha x)^{-2}\, dt = \text{const.}, \textit{ for } x \in J(a), \tag{3.2}$$

if and only if

$$2u(x) = (f(0))^{1/2} \frac{x-h(x)}{(1+\alpha x)(1+\alpha h(x))}, \textit{ for every } x \in J(a). \tag{3.3}$$

Remark that in the case of isochronism, i.e., (3.2) with $\alpha = 0$, formula (3.3) is particularly simple. Namely

$$2u(x) = (f(0))^{1/2}\,(x-h(x)), \textit{ for every } x \in J(a). \tag{3.3$'$}$$

COROLLARY 3. *We have*

$$\oint f(x)\, dt = \text{const.}, \textit{ for } x \in J(a), \tag{3.4}$$

if and only if

$$\frac{2}{u(x)}\,(f(0))^{1/2} = \frac{1}{x} - \frac{1}{h(x)}, \textit{ for every } x \in J(a)\setminus\{0\}. \tag{3.5}$$

Proof. By the definition of v (see $(2.1)_2$) and by $(2.3)_2$, we have

$$u(x)\,u'(x) = xf(x) \text{ for every } x \in J. \tag{3.6}$$

By this relation, and by Proposition 1 in Section 2, we have that (3.4) holds if and only if

$$2(f(0))^{1/2}\,u(x) = \int_{h(x)}^{x} \frac{u(\xi)\,u'(\xi)}{\xi}\, d\xi, \text{ for every } x \in J(a).$$

Thus the condition (3.4) is equivalent to (see (2.4))

$$2(f(0))^{1/2} z = \int_0^z \left(\frac{\eta}{X(\eta)} - \frac{\eta}{X(-\eta)}\right) d\eta, \text{ for every } z \in u(J(a)). \tag{3.7}$$

By taking the derivative of both members, and by passing to the variable

x again, we have that (3.4) implies (3.5). Furthermore, if (3.5) holds, then the integrand in (3.7) is equal to $2(f(0))^{1/2}$ on $u(J(a))\setminus\{0\}$. Thus (3.7) holds, and this implies (3.4). ▮

PROPOSITION 3. *Let $\alpha \in \mathbb{R}$. Both conditions* (3.2) *and* (3.4) *are satisfied if and only if*

$$f(x) = f(0)/(1+\alpha x)^3, \text{ for every } x \in J(a). \tag{3.8}$$

We call this function "the Keplerian map".

This result is an easy consequence of Corollaries 2, 3, and of (3.6), as the following Proposition.

PROPOSITION 4. *The condition in* (3.2) *is satisfied for some $\alpha \neq 0$ and also for $\alpha = 0$ (at the same time) if and only if*

$$f(x) = \frac{f(0)}{4} \frac{(2+\alpha x)(1+(1+\alpha x)^2)}{(1+\alpha x)^3}, \text{ for every } x \in J(a). \tag{3.9}$$

In Section 4 we shall see that the radial motion of the Isotropic Oscillator is related to (3.9).

4. MECHANICS

Let us forget for the moment the equation (1.4), and let us consider the celebrated problem of the motion of a point (of mass 1) in a Central Potential Field in ordinary space. The trajectory (in the 3-dimensional space) belongs to some plane which contains the centre of the field. Let us fix such a plane, and consider polar coordinates ρ, ϕ with $\rho = 0$ at the centre of the field. We have

$$\ddot{\rho} - \rho\dot{\phi}^2 = -U'(\rho), \tag{4.1}$$

where U is the Potential Energy. Furthermore, the conservation of the Angular Momentum, which is assumed nonvanishing in the sequel, yields

$$\rho^2\dot{\phi} = L \neq 0, \tag{4.2}$$

where L is a constant (determined by the initial conditions). By (4.2) we can get rid of $\dot{\phi}$ in (4.1). This yield the well known equation

$$\ddot{\rho} = -V_L'(\rho), \text{ where } V_L(\rho) = U(\rho) + L^2/2\rho^2. \tag{4.3}$$

The map V_L is called the "Effective Potential Energy".

In the sequel we fix $L \neq 0$ and we assume that

$$V'_L(\hat{\rho}) = 0, \; V''_L(\hat{\rho}) > 0, \text{ for some } \hat{\rho} > 0, \tag{4.4}$$

so that, in particular, the circular motion $\rho = \hat{\rho}$ is allowed. By setting

$$x = \rho - \hat{\rho} \text{ (so } x \text{ is also a radial coordinate)}, \tag{4.5}$$

the equation (4.3) can be written as

$$\ddot{x} + xf(x) = 0, \text{ with } f(0) > 0. \tag{4.6}$$

Now, let us define

$$\alpha = 1/\hat{\rho} \text{ (so, in particular, } \alpha > 0).$$

Then, by (4.2), we have that the condition in (3.2)

$$\oint (1 + \alpha x)^{-2} \, dt = \text{const., with } \alpha = 1/\hat{\rho},$$

is nothing but "isochronism with respect to the angle ϕ" (for $x \in J(a)$) (see the sentences from (1.5) to (1.6)). So, for the equation (4.6), i.e., (4.3) with (4.5), the condition in (3.2), with $\alpha = 1/\hat{\rho}$, has a direct physical meaning. In fact it requires that the difference between the values of the angle ϕ after one period of the radial coordinate, does not depend on the motion (notice that we fixed $L \neq 0$).

Of course this condition is satisfied for $U(\rho) = -1/\rho$, which is the potential energy of the Kepler's Problem, and any $L \neq 0$. In this case (4.4) holds (with $\hat{\rho} = L^2$). Moreover the map f is the map in (3.8) with $\alpha = 1/\hat{\rho}$ and $f(0) = 1/\hat{\rho}^3$. Thus, by Proposition 3, the condition (3.4) is also satisfied.

Consider now the potential energy of the Isotropic Oscillator, namely $U(\rho) = \rho^2/2$. For any $L \neq 0$, the conditions (4.4) hold (with $\hat{\rho} = (|L|)^{1/2}$). Since this dynamical problem has a trivial solution in cartesian coordinates, we do not need any calculations to say that the condition in (3.2) holds with $\alpha = 0$ and with $\alpha = 1/\hat{\rho}$ too. This agrees with Proposition 4 in Section 3 because, in this case f is the (nontrivial) map in (3.9) with $\alpha = 1/\hat{\rho}$ and $f(0) = 4$.

5. Liapunov Stability

Proposition 5. *Let f be a C^0 real map defined in some neighbourhood of 0 in $\mathbb{R}$. Then the origin is a stable equilibrium for the system*

$$\ddot{x} + xf(x) = 0, \; \ddot{y} + yf(x) = 0, \tag{5.1}$$

if and only if $f(0)>0$ and the origin is (locally) a f-isochronous center for the scalar equation $(5.1)_1$ (see Def. 1 in Sect. 2).

Remark that we can consider $f(0)=1$ instead of $f(0)>0$ (in fact we just have to perform the transformation $t \mapsto (f(0))^{1/2}\, t$). Then Proposition 5 comes from Corollary 3 in Section 3, and from the Corollary in Section 4 of the paper [16] (where $f(0)=1$ was assumed).

PROPOSITION 6. *Let f be as in Proposition 5, but $f \in C^1$, and $\alpha \in \mathbb{R}$. Then the origin is a stable equilibrium for the system*

$$\ddot{x}+xf(x)=0,\ \ddot{y}+y\,\frac{f(x)+xf'(x)+\alpha x(4f(x)+xf'(x))}{1+\alpha f(x)}=0, \tag{5.2}$$

if and only if $f(0)>0$ and the origin is (locally) a $(1+ax)^{-2}$-isochronous center for the equation $(5.2)_1$.

(We used the notation $(1+\alpha x)^{-2}$ for the map $x \mapsto (1+\alpha x)^{-2}$).

This Proposition can be obtained from the Proposition in Section 4 of the paper [18] (where $f(0)=1$ is considered) and from the Corollary 2 that we proved in Section 3.

Let us end this Section with the following Remark (which is not used in the sequel).

Remark. Let f be as in Proposition 5, let $f(0)>0$ and consider $\ddot{x}+xf(x)=0$. Then $\oint dt = \text{const.}$, for $x \in J(a)$, if and only if every solution $x(\cdot)$ with $x(0) \in J(a)$ and $\dot{x}(0)=0$ is Liapunov stable. (Here, of course, a is arbitrary but $a \in J$ and $a>0$).

This Remark is proved by the continuity of the flow.

6. CONSTRUCTIONS

In the paper [16], the author constructs, in two different ways, the collection of all the continuous maps f, with $f(0)=1$, which yield stability of the origin for the system (5.1) (see Sect. 4 of [16] after the Corollary). The general case, i.e., for $f(0)>0$ not necessarily equals to 1, can be obtained by the trivial multiplication of the aforementioned maps by strictly positive constants.

The first construction consists in extending via the equality in condition (3.5), any given C^0 map $\bar{f}\colon [0, a[\, \to \mathbb{R}$, with $\bar{f}(0)>0$, $a>0$, to some neighbourhood of 0 (see Sect. 4 of [16]).

The second construction is obtained by observing that any $h\colon J \to J$ which satisfies (2.6), where J is an open interval containing 0, the equality in condition (3.5), and (3.6), determine the typical map f which yields stability. Thus we just have to construct the typical h which satisfies (2.6). This is done by a $\pi/4$-rotation of the graph of any even C^1 map (see Remark 1 in Sect. 3 of [16] and the end of Sect. 4 in [16]).

By Proposition 5, in this way we gave constructions of the continuous maps f, with $f(0)>0$, such that the origin is (locally) a f-isochronous center for

$$\ddot{x}+xf(x)=0,\ f(0)>0. \tag{6.1}$$

We can use Proposition 6 (in this paper), and Section 5 of [18] to give similar constructions for the C^1 maps f such that the origin is a $(1+\alpha x)^{-2}$-isochronous center for (6.1) (locally). But it is also easy to extend these constructions to the C^0 case (as a matter of fact C^0 is easier than C^1 in these arguments).

Let us end this Section with a geometrical construction in the particular case of (ordinary) isochronism. We deal with the potential energy v (see $(2.1)_2$).

By Corollary 2 in Section 3, it is easy to see that for (6.1):

$$\oint dt=\text{const., for } x\in J(a) \text{ iff}$$

$$v(x)=\frac{f(0)}{2}\left(\frac{x-h(x)}{2}\right)^2 \text{ for every } x\in J(a). \tag{6.2}$$

(Of course $a>0$, and $a\in J$). The potential energy v in (6.2) has a nice geometrical property.

Let us consider the parabolas

$$P_c^\lambda=\{(x,\zeta)\in\mathbb{R}^2 : 2\zeta=\lambda(x-c)^2\},\ c\in\mathbb{R},\ \lambda\in\mathbb{R}_+^*. \tag{6.3}$$

If the condition in (6.2) is satisfied, then, for any $\hat{x}\in\,]0,a[$, the points $(\hat{x},v(\hat{x}))$ and $(h(\hat{x}),v(h(\hat{x})))$ belong to the parabola P_c^λ in (6.3) with $\lambda=f(0)$ and $2c=\hat{x}+h(\hat{x})$. This parabola has its vertex at the center of the segment of the x-axis having $h(\hat{x})$ and $\hat{x}$ as endpoints.

Furthermore the family $\{P_c^\lambda : c\in\mathbb{R}\}$ is produced by the translation of a single parabola whose vertex lies on the x-axis and whose axis is parallel to the ζ-axis (see (6.3)).

Therefore we can imagine an ideal "pantograph" for the graphs of the potential energies of the isochronous centers.

We can consider the graph, in the x, ζ-plane of any $\bar{v}\in C^1([0,b[;\mathbb{R}_+^*)$, where $b>0$, which satisfies $\bar{v}(0)=0$, $\bar{v}'(0)=0$ and admits $\bar{v}''(0)>0$. Then we can construct the graph of the potential energy of an isochronous center from this curve by intersecting it with the aforementioned parabolas with $\lambda = \bar{v}''(0)$ (see (2.2)). For any given $\hat{x} \in]0, b[$ we consider the parabola passing through $(\hat{x}, v(\hat{x}))$. Then we mark the other point on the intersection of this parabola with the line $\zeta = v(\hat{x})$ (which is parallel to the x-axis). As $\hat{x}$ varies, we so extend the graph of $\bar{v}$ to a curve in the x, ζ-plane. After a suitable restriction of this curve we obtain the graph of the potential energy of a (globally) isochronous center.

7. Oscillations with Prescribed Periods

In this section we do not assume (1.4). We start from an arbitrary map $\bar{f} \in C^0([0, b[; \mathbb{R}_+^*)$, where $0 < b \leqslant +\infty$, and from an arbitrary map $\bar{\ell} \in C^1([0, b[; \mathbb{R}_+^*)$, with $\bar{\ell}(0) = 2\pi/(\bar{f}(0))^{1/2}$. Then we constructively prove the existence of a unique maximal map $f \in C^0(A; \mathbb{R}_+^*)$, with $0 \in A = A^0$ and $A \cap \mathbb{R}_+^* \subset [0, b[$, such that:

(i) $f(x) = \bar{f}(x)$ for every $x \in A \cap \mathbb{R}_+$;

(ii) the interval J associated with f (as in Sect. 2) coincides with A, that is the origin is a global center for $\ddot{x} + xf(x) = 0$;

(iii) for any $x \in A \setminus \{0\}$, the period of the orbit $\gamma(x, 0)$ of $\ddot{x} + xf(x) = 0$ through $(x, 0)$, coincides with $\bar{\ell}(x)$.

(Thus, in particular, the value of $\bar{\ell}(0)$ above agrees with Prop. 2).

However, let us remark that this construction does not give all the continuous f yielding a global center, because the aforementioned map $\bar{\ell}$ is C^1 while the map in Proposition 2 (which corresponds to $m = 1$) is just C^0 in general.

We first consider the maps $\bar{v}$ and $\bar{u}$ defined on $[0, b[$ by

$$\bar{v}(x) = \int_0^x \xi \bar{f}(\xi)\, d\xi, \text{ and } \bar{u}(x) = (2\bar{v}(x))^{1/2}.$$

The map $\bar{u}$ is C^1 and $\bar{u}'(x) > 0$ (since $\bar{f}(x) > 0$) for every $x \in [0, b[$. Therefore it is a C^1 diffeomorphism onto some $[0, c[$. Let $\bar{X}$ be its inverse and let $\bar{T} = \bar{\ell} \circ \bar{X}$. Now, let us consider the extension $\tilde{X}$ of $\bar{X}$ to $]-c, c[$ obtained by

$$\tilde{X}(-z) = \bar{X}(z) - \pi^{-1} \int_0^z \theta \bar{T}(\theta)(z^2 - \theta^2)^{-1/2}\, d\theta, \text{ for } z \in]0, c[, \qquad (7.1)$$

(see (2.14) with $M = 1$). The change of variables $\theta \to \sin^{-1}(\theta/z)$ shows that $\tilde{X}$ is C^1 (because so are $\bar{X}$ and $\bar{T}$). Furthermore $\tilde{X}'(0) > 0$. Therefore we can consider the greatest positive number $d\,(\leqslant c)$ such that X, defined as

$\tilde{X} \mid]-d, d[$, is a C^1 diffeomorphism onto some open interval J which contains 0. Let $u = X^{-1}$ and finally let us define $f: J \to \mathbb{R}_+^*$ by $f(x) = u(x)\, u'(x)/x$ for $x \neq 0$ and by $f(0) = \bar{f}(0)$.

Now (i) and (ii) above are obvious, as well as the other properties of f enunciated above except (iii). In order to prove (iii) let us first remark that

$$\int_0^z \theta(z^2 - \theta^2)^{-1/2}\, \bar{T}(\theta)\, d\theta$$

$$= -(z^2 - \theta^2)^{1/2}\, \bar{T}(\theta)\big|_{\theta=0}^{\theta=z} + \int_0^z (z^2 - \eta^2)^{1/2}\, \bar{T}'(\eta)\, d\eta.$$

Therefore, by (7.1), we have that

$$\pi(X'(z) + X'(-z)) = \bar{T}(0) + z \int_0^z (z^2 - \eta^2)^{-1/2}\, \bar{T}'(\eta)\, d\eta$$

for every $z \in]0, d[$. Now, replace z by θ, multiply by $(z^2 - \theta^2)^{-1/2}$ and integrate with respect to θ from 0 to z. Then we easily have

$$\bar{T}(z) = 2 \int_0^z (z^2 - \theta^2)^{-1/2}\, (X'(\theta) + X'(-\theta))\, d\theta, \text{ for every } z \in]0, d[$$

(see the proof of Prop. 1 in Sect. 2). And finally

$$\bar{\ell}(x) = 2 \int_{h(x)}^x (2(v(x) - v(\xi)))^{-1/2}\, d\xi, \text{ for every } x \in J,\ x > 0,$$

(where v and h are associated with f as in Sect. 2). By the first integral in (2.1), the sentence (iii) above is proved. ∎

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References

1. A. Barone-Netto and M. O. Cesar, Nonconservative Positional Systems-Stability, *Dyn. and Stability of Systems* **2** (4) (1988), 213–221.
2. I. Koukles and N. Piskounov, Sur les vibrations tautochrones dans les systèmes conservatifs et non conservatifs, *C. R. Acad. Sci. URRS* **17** (1937), 471–475.

3. L. D. LANDAU AND E. M. LIFSHITZ, "Mechanics", Course of Theoretical Physics, Vol. 1, Pergamon Press, Oxford, 1960.
4. J. J. LEVIN AND S. S. SHATZ, Nonlinear oscillations of fixed period, *J. Math. Anal. Appl.* **7** (1963), 284–288.
5. W. MAGNUS AND S. WINKLER, "Hill's Equation", Interscience, New York (1966).
6. C. OBI, Analytical theory of nonlinear oscillations. VII. The periods of the periodic solutions of the equation $\ddot{x}+g(x)=0$, *J. Math. Anal. Appl.* **55** (1976), 295–301.
7. C. OBI, Analytical theory of nonlinear oscillations. VIII. Second-order conservative systems whose solutions are all oscillating with the least period 2π, *Ann. Mat. Pura Appl. (4)* **117** (1978), 339–347.
8. Z. OPIAL, Sur les périodes des solutions de l'équation différentielle $x''+g(x)=0$, *Ann. Polon. Math.* **10** (1961), 49–72.
9. L. TONELLI, Su un problema di Abel, *Math. Annalen* **99** (1928), 183–199.
10. M. URABE, Potential forces which yield periodic motions of fixed period", *J. Math. Mech.* **10** (1961), 569–578.
11. M. URABE, The potential force yielding a periodic motion whose period is an arbitrary continuous function of the amplitude of the velocity, *Arch. Rat. Mech. Anal.* **11** (1962), 27–33.
12. M. URABE, The potential force yielding a periodic motion whose period is an arbitrary continuously differentiable function of the amplitude, *J. Sci. Hiroshima Univ. Ser. A–I* **26** (1962), 93–109.
13. M. URABE, The potential force yielding a periodic motion with arbitrary continuous half-periods, *J. Sci. Hiroshima Univ. Ser. A–I* **26** (1962), 111–122.
14. M. URABE, Relations between periods and amplitudes of periodic solutions of $\ddot{x}+g(x)=0$, *Funkcialaj Ekvacioj* **6** (1964), 63–88.
15. G. ZAMPIERI AND A. BARONE-NETTO, "Attractive Central Forces may yield Liapunov Instability", Proceed. VII Elam, Editorial Equinoccio, Caracas (1986), 105–112.
16. G. ZAMPIERI, Liapunov stability for some central forces, *J. Differential Equations* **74** (1988), 254–265.
17. G. ZAMPIERI, "Some problems in Stability of the Equilibrium related to Coexistence of Solutions to Hill's Equations", to appear in the Proceed. 1987 Equadiff., Lecture Notes in Pure and Applied Mathematics, Dekker.
18. G. ZAMPIERI, Solving a collection of free coexistence-like problems in stability, to appear in *Rend. Sem. Mat. Univ. Padova.*